Möbius Conjugation and Convolution Formulae

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Abstract

Let P be a locally finite poset with the interval space $\mathrm{Int}(P)$, and R a ring with identity. We shall introduce the Möbius conjugation μ^* sending each function $f: P \to R$ to an incidence function $\mu^*(f): \mathrm{Int}(P) \to R$ such that $\mu^*(fg) = \mu^*(f) * \mu^*(g)$. Taking P to be the intersection poset of a hyperplane arrangement $\mathcal{A}$, we shall obtain a convolution identity for the number $r(\mathcal{A})$ of regions and the number $b(\mathcal{A})$ of relatively bounded regions, and a reciprocity theorem of the characteristic polynomial $\chi(\mathcal{A}, t)$, which also leads to a combinatorial interpretation to the values $|\chi(\mathcal{A}, -q)|$ for large primes q. Moreover, all known convolution identities on Tutte polynomials of matroids will be direct consequences after specializing the poset P and functions f, g.

Keywords: Möbius conjugation, convolution formula, reciprocity theorem, characteristic polynomial, Tutte polynomial, hyperplane arrangement, matroids

1 Möbius Conjugation

We use the definitions and notations of posets from [7], and all posets in this paper are assumed to be locally finite. Let P be a poset and R a ring with identity. Denote by $\mathrm{Int}(P)$ the interval space of P and $\mathcal{I}(P, R) = \{\alpha : \mathrm{Int}(P) \to R\}$ the incidence algebra of P whose multiplication structure is given by the convolution product, i.e., for any $\alpha, \beta \in \mathcal{I}(P)$ and $x \le y$ in P,

$$[\alpha * \beta](x, y) = \sum_{x \le z \le y} \alpha(x, z)\, \beta(z, y), \quad \forall\, \alpha, \beta \in \mathcal{I}(P).$$

Let R^P be the ring of all functions $f : P \to R$ whose ring structure is given by point-wise multiplication and addition. Define the *Möbius conjugation* $\mu^* : R^P \to \mathcal{I}(P, R)$ to be

$$\mu^*(f) = \mu * \delta(f) * \zeta, \quad \forall\, f \in R^P.$$

where μ is the Möbius function of P, and the map $\delta : R^P \to \mathcal{I}(P, R)$ is defined by $\delta(f)(x, y) = f(x)$ if $x = y$ and 0 otherwise, for all $f \in R^P$ and $x \le y$ in P.

Theorem 1.1. *With above settings, the map μ^* is a ring monomorphism, i.e.,*

$$\mu^*(fg) = \mu^*(f) * \mu^*(g), \quad \forall\, f, g \in R^P.$$

Proof. Given $f, g \in R^P$, it is obvious that $\delta(fg) = \delta(f) * \delta(g)$. Then

$$\mu^*(fg) = \mu * \delta(f) * \zeta * \mu * \delta(g) * \zeta = \mu^*(f) * \mu^*(g).$$

So μ^* is a homomorphism as rings. To prove the injectivity, suppose $\mu^*(f) = \mu^*(g)$ for some $f, g \in R^P$. Multiplying ζ on the left hand side and μ on the right hand side respectively, we obtain that $\delta(f) = \delta(g)$. Thus $f = g$. □

Multiplicative identities for chromatic polynomials first appeared in [10] by W. Tutte in 1967. In 1999, W. Kook, V. Riener and D. Stanton [4] found a convolution formula for Tutte polynomial of matroids. After that, Joseph P. S. Kung [5] gave a multiplicative identities for characteristic polynomials of matroids in 2004. And he also formulated many generalizations of all previous identities in 2010 [6]. We shall see that Theorem 1.1 gives the algebraic essence to all these known identities. In fact, by specializing the poset P and the functions f, g of Theorem 1.1, we will obtain all those identities.

2 Convolution Formula on Characteristic Polynomials

A hyperplane arrangement $\mathcal{A}$ in a vector space V is a collection of finite hyperplanes of V. The intersection semi-lattice $L(\mathcal{A})$ of $\mathcal{A}$ is defined to be the collection of all nonempty intersections of hyperplanes in $\mathcal{A}$, whose partial order is given by the inverse of set inclusion. Namely,

$$L(\mathcal{A}) = \{\cap_{H\in\mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A}\},$$

whose minimal element is $\hat{0} = \cap_{H\in\emptyset} H := V \in L(\mathcal{A})$. Artificially adding a maximal element $\hat{1} = \emptyset$ to $L(\mathcal{A})$, $L(\mathcal{A})$ then becomes a geometric lattice, denoted $L^*(\mathcal{A}) = L(\mathcal{A}) \cup \{\hat{1}\}$ and called the *reduced intersection lattice* of $\mathcal{A}$. With the assumptions $\dim(\hat{1}) = \infty$ and $t^\infty = 0$, the *characteristic polynomial* $\chi(\mathcal{A}, t) \in \mathbb{C}[t]$ of $\mathcal{A}$ can be written as

$$\chi(\mathcal{A}, t) := \sum_{X\in L(\mathcal{A})} \mu(\hat{0}, X)\, t^{\dim(X)} = \sum_{X\in L^*(\mathcal{A})} \mu(\hat{0}, X)\, t^{\dim(X)}.$$

Given $X, Y \in L^*(\mathcal{A})$ and $X \le Y$, let $\mathcal{A}_{X,Y}$ be a hyperplane arrangement in the vector space X defined by

$$\mathcal{A}_{X,Y} = \{H \cap X \mid H \in \mathcal{A} \text{ with } Y \subseteq H \text{ and } X \not\subseteq H\}.$$

In particular, $\mathcal{A}_{X,X} = \emptyset$. If $X, Y \in L(\mathcal{A})$ and $X \le Y$, it is easily seen that $L(\mathcal{A}_{X,Y}) \cong [X, Y]$ as lattices. It follows that the Möbius function of $L^*(\mathcal{A}_{X,Y})$ is the same as the restriction of the Möbius function of $L^*(\mathcal{A})$ onto the interval $[X, Y]$. Then the characteristic polynomial of $\mathcal{A}_{X,Y}$ is

$$\chi(\mathcal{A}_{X,Y}, t) = \sum_{X\le Z\le Y} \mu(X, Z)\, t^{\dim(Z)},$$

where μ is the Möbius function of $L^*(\mathcal{A})$. In particular, $\chi(\mathcal{A}_{\hat{0},\hat{1}}, t) = \chi(\mathcal{A}, t)$ and $\chi(\mathcal{A}_{\hat{1},\hat{1}}, t) = 0$. For convenience, we denote, for any $X \in L(\mathcal{A})$,

$$\begin{aligned} \mathcal{A}|X &= \mathcal{A}_{\hat{0},X} = \{H \in \mathcal{A} \mid X \subseteq H\}, \\ \mathcal{A}/X &= \mathcal{A}_{X,\hat{1}} = \{H \cap X \mid H \in \mathcal{A} - \mathcal{A}|X\}. \end{aligned}$$

Theorem 2.1. *Let $\mathcal{A}$ be a hyperplane arrangement with the reduced intersection lattice $L^*(\mathcal{A})$. If $X \le Y$ in $L^*(\mathcal{A})$, then we have*

$$\chi(\mathcal{A}_{X,Y}, st) = \sum_{Z\in L^*(\mathcal{A}); X\le Z\le Y} \chi(\mathcal{A}_{X,Z}, s)\, \chi(\mathcal{A}_{Z,Y}, t).$$

Taking $X = \hat{0}$ and $Y = \hat{1}$ in particular, we have

$$\chi(\mathcal{A}, st) = \sum_{X\in L(\mathcal{A})} \chi(\mathcal{A}|X, s)\, \chi(\mathcal{A}/X, t). \tag{1}$$

Proof. Define $f, g : L^*(\mathcal{A}) \to \mathbb{C}[s, t]$ to be $f(X) = t^{\dim(X)}$ and $g(X) = s^{\dim(X)}$. Then for any $X \le Y$ in $^*(\mathcal{A})$, we have

$$\mu^*(f)(X, Y) = [\mu * \delta(f) * \zeta]\,(X, Y) = \sum_{X\le Z\le Y} \mu(X, Z) t^{\dim(Z)} = \chi(\mathcal{A}_{X,Y}, t).$$

Similarly, $\chi(\mathcal{A}_{X,Y}, s) = \mu^*(g)(X, Y)$ and $\chi(\mathcal{A}_{X,Y}, st) = \mu^*(gf)(X, Y)$. Applying Theorem 1.1, then

$$\chi(\mathcal{A}_{X,Y}, st) = [\mu^*(g) * \mu^*(f)]\,(X, Y) = \sum_{Z\in L^*(\mathcal{A}); X\le Z\le Y} \mu^*(g)(X, Z)\mu^*(f)(Z, Y).$$

This completes the proof. ∎

Joseph P. S. Kung [5] found the convolution formula (1) for characteristic polynomials of matroids. Here the formula (1) of Theorem 2.1 extends it to affine hyperplane arrangements, which are not necessarily a matroid. In the next two subsections, we shall give two combinatorial identities by applying the formula (1).

2.1 Convolution Formula on $r(\mathcal{A})$ and $b(\mathcal{A})$

If $\mathcal{A}$ is a hyperplane arrangement in the real vector space $V = \mathbb{R}^n$, its complement $M(\mathcal{A}) = V - \cup_{H\in\mathcal{A}} H$ consists of finite many disjoint connected components, called *regions* of $\mathcal{A}$. If $\dim(V) = n$, denote by $\mathscr{R}(\mathcal{A})$ the collection of all regions of $M(\mathcal{A})$ and $r(\mathcal{A}) = \#\mathscr{R}(\mathcal{A})$. Let W be the subspace spanned by the normal vectors of H for all $H \in \mathcal{A}$. A region $\Delta \in \mathscr{R}(\mathcal{A})$ is called *relatively bounded* if $\Delta \cap W$ is bounded in W. Denote by $\mathscr{B}(\mathcal{A})$ the collection of all relatively bounded regions of $M(\mathcal{A})$ and $b(\mathcal{A}) = \#\mathscr{B}(\mathcal{A})$. Zaslavski formula [11] states that

$$r(\mathcal{A}) = (-1)^{\dim(V)}\chi(\mathcal{A}, -1), \quad b(\mathcal{A}) = (-1)^{\operatorname{rank}(\mathcal{A})}\chi(\mathcal{A}, 1),$$

where $\operatorname{rank}(\mathcal{A}) = \max\{\operatorname{rank}(X) \mid X \in L(\mathcal{A})\}$ and $\operatorname{rank}(X) = \dim(V) - \dim(X)$. Then the following convolution formula of $r(\mathcal{A})$ and $b(\mathcal{A})$ can be easily obtained from (1).

Theorem 2.2. *Denote by* $\operatorname{corank}(X) = \operatorname{rank}(\mathcal{A}) - \operatorname{rank}(X)$. *Then*

$$\begin{aligned} b(\mathcal{A}) &= \sum_{X\in L(\mathcal{A})} (-1)^{\operatorname{corank}(X)}\, r(\mathcal{A}|X)\, r(\mathcal{A}/X), \\ r(\mathcal{A}) &= \sum_{X\in L(\mathcal{A})} (-1)^{\operatorname{corank}(X)}\, r(\mathcal{A}|X)\, b(\mathcal{A}/X). \end{aligned}$$

We next give a combinatorial proof to above identities. The idea is to consider the total signs on the right side contributed to each region on the left side of the identities. If $\mathcal{A}$ is a hyperplane arrangement in $\mathbb{R}^n$, recall that W is the subspace spanned by the normal vectors of all hyperplanes in $\mathcal{A}$. Consider the arrangement $\mathcal{A}_W = \{H \cap W \mid H \in \mathcal{A}\}$ in W whose rank $r(\mathcal{A}_W) = \dim(W)$. It is easily seen that $r(\mathcal{A}) = r(\mathcal{A}_W)$, $b(\mathcal{A}) = b(\mathcal{A}_W)$, and $L(\mathcal{A}) \cong L(\mathcal{A}_W)$. Assume that $\mathcal{A}$ is a hyperplane arrangement in $\mathbb{R}^n$ with rank $r(\mathcal{A}) = n$. Then all relatively bounded regions in $M(\mathcal{A})$ are actually bounded and $\operatorname{corank}(X) = \dim(X)$ for all $X \in L(\mathcal{A})$. With the induced topology of the standard topology of $\mathbb{R}^n$, each region $\Delta \in \mathscr{R}(\mathcal{A})$ is homeomorphic to an open ball of dimension n. Given any $\Delta \in \mathscr{R}(\mathcal{A})$, let $\bar{\Delta}$ be its topological closure and denote by $F(\Delta)$ the collection of all *faces* of $\bar{\Delta}$, which is defined to be

$$F(\Delta) = \{\bar{\Delta} \cap M(\mathcal{A}/X) \mid X \in L(\mathcal{A}), \bar{\Delta} \cap M(\mathcal{A}/X) \neq \emptyset\}.$$

If $\Delta \in \mathscr{B}(\mathcal{A})$ is a bounded region in $M(\mathcal{A})$, $\bar{\Delta}$ becomes a closed polytope and is homeomorphic to a closed ball. Then the Euler characteristic of $\bar{\Delta}$ is 1, i.e.,

$$\sum_{f\in F(\Delta)} (-1)^{\dim(f)} = 1, \quad \forall\, \Delta \in \mathscr{B}(\mathcal{A}). \tag{2}$$

If $\Delta \in \mathscr{R}(\mathcal{A}) - \mathscr{B}(\mathcal{A})$ is an unbounded region in $M(\mathcal{A})$, denote by $F_b(\Delta)$ the collection of bounded faces of $\bar{\Delta}$. Then the space $\sqcup_{f\in F_b(\Delta)} f$ is homeomorphic to a closed ball, i.e.,

$$\sum_{f\in F_b(\Delta)} (-1)^{\dim(f)} = 1, \quad \forall\, \Delta \in \mathscr{R}(\mathcal{A}) - \mathscr{B}(\mathcal{A}). \tag{3}$$

On the other hand, when $\Delta \in \mathscr{R}(\mathcal{A}) - \mathscr{B}(\mathcal{A})$ is an unbounded region in $M(\mathcal{A})$, $\bar{\Delta}$ is homeomorphic to a closed half space whose Euler characteristic is 0. Then we have

$$\sum_{f\in F(\Delta)} (-1)^{\dim(f)} = 0. \tag{4}$$

Applying (2), (3), and (4), we have

$$r(\mathcal{A}) = \sum_{\Delta\in\mathscr{B}(\mathcal{A})} \sum_{f\in F(\Delta)} (-1)^{\dim(f)} + \sum_{\Delta\in\mathscr{R}(\mathcal{A})-\mathscr{B}(\mathcal{A})} \sum_{f\in F_b(\Delta)} (-1)^{\dim(f)} = \sum_{\Delta\in\mathscr{R}(\mathcal{A})} \sum_{f\in F_b(\Delta)} (-1)^{\dim(f)},$$

$$b(\mathcal{A}) = \sum_{\Delta\in\mathscr{B}(\mathcal{A})}\sum_{f\in F(\Delta)}(-1)^{\dim(f)} + \sum_{\Delta\in\mathscr{R}(\mathcal{A})-\mathscr{B}(\mathcal{A})}\sum_{f\in F(\Delta)}(-1)^{\dim(f)} = \sum_{\Delta\in\mathscr{R}(\mathcal{A})}\sum_{f\in F(\Delta)}(-1)^{\dim(f)}.$$

From the definition of $F(\Delta)$, we can see that for each face $f \in F(\Delta)$, there exists a unique $X \in L(\mathcal{A})$ such that $f \subseteq X$ and $\dim(f) = \dim(X)$, i.e., $f \in \mathscr{R}(\mathcal{A}/X)$. However, for each $f \in \mathscr{R}(\mathcal{A}/X)$, there are $r(\mathcal{A}|X)$ regions $\Delta \in \mathscr{R}(\mathcal{A})$ such that $f \in F(\Delta)$. In addition, f is bounded if and only if $f \in \mathscr{B}(\mathcal{A}/X)$. Then we have

$$\begin{aligned} r(\mathcal{A}) &= \sum_{X\in L(\mathcal{A})}(-1)^{\dim(X)}\, r(\mathcal{A}|X)\, b(\mathcal{A}/X),\\ b(\mathcal{A}) &= \sum_{X\in L(\mathcal{A})}(-1)^{\dim(X)}\, r(\mathcal{A}|X)\, r(\mathcal{A}/X). \end{aligned}$$

According to the assumption that $\mathcal{A}$ is essential in $\mathbb{R}^n$, we have $\operatorname{corank}(X) = \dim(X)$ for all $X \in L(\mathcal{A})$ which completes the proof.

2.2 Reciprocity Theorem of Characteristic polynomials

In this subsection, the hyperplane arrangement $\mathcal{A}$ is assumed to be *integral*, i.e., each hyperplane $H \in \mathcal{A}$ is defined by an integral linear equation as follows

$$H:\ a_1x_1 + \cdots + a_nx_n = b, \quad b, a_i \in \mathbb{Z},\ 1 \le i \le n. \tag{5}$$

For any prime number q, the above hyperplane H is automatically reduced to a hyperplane H_q in $\mathbb{F}_q^n$, which is defined by the equation

$$H_q:\ a_1x_1 + \cdots + a_nx_n = b \mod q,$$

called the q-reduction of H. Then $\mathcal{A}_q = \{H_q \mid H \in \mathcal{A}\}$ defines a hyperplane arrangement in the space $V = \mathbb{F}_q^n$. In this subsection, we always assume q is large enough such that that the intersection semilattices $L(\mathcal{A})$ and $L(\mathcal{A}_q)$ are isomorphic, where the isomorphism is given by

$$X \mapsto X_q := \cap_{H\in\mathcal{A}|X} H_q.$$

Similar as before, we have the following notations,

$$\mathcal{A}_q|X_q = \{H_q \mid H \in \mathcal{A}|X\}, \quad \mathcal{A}_q/X_q = \{H_q \cap X_q \mid H \in \mathcal{A}/X\}, \quad M(\mathcal{A}_q) = \mathbb{F}_q^n - \cup_{H\in\mathcal{A}} H_q.$$

C. A. Athanasiadis gave the following combinatorial interpretation of $\chi(\mathcal{A}, q)$.

Theorem 2.3. [1] *Let $\mathcal{A}$ be an integral arrangement in $\mathbb{R}^n$ and q be a large prime number. Then*

$$\chi(\mathcal{A}, q) = \big|M(\mathcal{A}_q)\big|.$$

It follows by (1) that $\chi(\mathcal{A}, -q) = \sum_{X\in L(\mathcal{A})} \chi(\mathcal{A}|X, -1)\, \chi(\mathcal{A}/X, q)$. Applying Zaslavski formula $r(\mathcal{A}|X) = (-1)^n\chi(\mathcal{A}|X, -1)$, we then obtain the following result which will leads to a combinatorial interpretation to the number $\chi(\mathcal{A}, -q)$ for any large prime number q, known as the reciprocity theorem of the characteristic polynomial.

Proposition 2.4. *With the same assumptions as* Theorem 2.3*, we have*

$$\chi(\mathcal{A}, -q) = (-1)^n \sum_{X\in L(\mathcal{A})} r(\mathcal{A}|X)\,\big|M(\mathcal{A}_q/X_q)\big|.$$

To state the combinatorial aspect of above formula, we introduce some notations first. Fixing a large prime number q, denote by

$$C(n, q) = \left\{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid -\frac{q-1}{2} \le x_i \le \frac{q-1}{2},\ \forall\, 1 \le i \le n\right\}$$

the central symmetric lattice cube of size q and dimension n. Suppose $\mathcal{A}$ is an integral hyperplane arrangement and $H \in \mathcal{A}$ is defined by the equation (5). For any $t \in \mathbb{R}$, let $H(t)$ be a hyperplane translated from H, whose defining equation is

$$H(t): \quad a_1x_1 + \cdots + a_nx_n = b + t.$$

For any large prime number q, denote by $M(\mathcal{A}, q)$ the complement of the union of $H(kq)$ for all $H \in \mathcal{A}$ and $k \in \mathbb{Z}$, and $\mathscr{R}(\mathcal{A}, q)$ the collection of all connected components (regions) of $M(\mathcal{A}, q)$, i.e.,

$$M(\mathcal{A}, q) = \mathbb{R}^n - \cup_{H\in\mathcal{A},k\in\mathbb{Z}} H(kq), \quad \mathscr{R}(\mathcal{A}, q) = \{\Delta \mid \Delta \text{ is a region of } M(\mathcal{A}, q)\}.$$

Now we are ready to state the reciprocity theorem for the characteristic polynomial of hyperplane arrangements, where the proof will be given later.

Theorem 2.5. [Reciprocity Theorem] *Let $\mathcal{A}$ be an integral arrangement with the characteristic polynomial $\chi(\mathcal{A}, t)$. For any large prime q and $\Delta \in \mathscr{R}(\mathcal{A}, q)$, denote by $\bar{\Delta}$ the topological closure of Δ and define*

$$\bar{\chi}(\mathcal{A}, q) := \sum_{\Delta\in\mathscr{R}(\mathcal{A},q)} \big|\bar{\Delta} \cap C(n, q)\big|.$$

Then $\bar{\chi}(\mathcal{A}, q) = (-1)^n\chi(\mathcal{A}, -q)$. Namely, $|\chi(\mathcal{A}, -q)|$ counts the total number of lattice points in $\bar{\Delta} \cap C(n, q)$ for all $\Delta \in \mathscr{R}(\mathcal{A}, q)$.

It should be noted that some other reciprocity theorems concerning characteristic polynomials of hyperplane arrangements have been studied. Stanley [9] introduced a reciprocity law for the chromatic polynomials of graphs. Athanasiadis [2] found a reciprocity law for the characteristic polynomial of a deformed linear arrangement, whose specialization on $m = 1$ provides a different interpretation to $\chi(\mathcal{A}, -q)$. As a direct consequence of standard Ehrhart theory, Beck and Zaslavski [3] generalized the reciprocity law of Ehrhart quasi-polynomials to a convex polytope dissected by a hyperplane arrangement. We shall see how Theorem 2.5 is related to Beck and Zaslavsky's reciprocity theorem, and then connected to the Ehrhart theory in the next paragraph.

Assume that $\mathcal{A}$ is a linear arrangement, i.e., the defining equations of all hyperplanes in $\mathcal{A}$ are homogeneous. With this assumption, denote by $\mathcal{A}^* = \{H(k) \mid H \in \mathcal{A}, k \in \mathbb{Z}\}$ the deformed arrangement of $\mathcal{A}$. Let $P = (-\frac{1}{2}, \frac{1}{2})^n \subseteq \mathbb{R}^n$. Then $P \setminus \bigcup_{H\in\mathcal{A},k\in\mathbb{Z}} H(k)$ consists of finite many open rational polytopes, denoted by $R_1, \ldots, R_l$. Let

$$E_{P,\mathcal{A}}(q) = \sum_{i=1}^{l} |qR_i \cap \mathbb{Z}^n|, \quad \bar{E}_{P,\mathcal{A}}(q) = \sum_{i=1}^{l} |q\bar{R}_i \cap \mathbb{Z}^n|.$$

Note that for all $x \in \mathbb{Z}^n$, $q^{-1}x \in H(k) \Leftrightarrow x \in H(kq)$, and $q^{-1}x \in P \Leftrightarrow x \in C(n, q)$. It follows that

$$E_{P,\mathcal{A}}(q) = \sum_{\Delta\in\mathscr{R}(\mathcal{A},q)} \big|\Delta \cap C(n, q)\big|, \quad \bar{E}_{P,\mathcal{A}}(q) = \sum_{\Delta\in\mathscr{R}(\mathcal{A},q)} \big|\bar{\Delta} \cap C(n, q)\big|\bar{\chi}(\mathcal{A}, q).$$

From Theorem 2.3, it is easily seen that $E_{P,\mathcal{A}}(q) = \chi(\mathcal{A}, q)$. Beck and Zaslavsky's reciprocity theorem [3] states that $\bar{E}_{P,\mathcal{A}}(q) = (-1)^nE_{P,\mathcal{A}}(-q)$. It follows that $\bar{\chi}(\mathcal{A}, q) = (-1)^n\chi(\mathcal{A}, q)$, which completes the proof of Theorem 2.5 in the case that $\mathcal{A}$ is linear. We know that Beck and Zaslavsky's reciprocity theorem is a direct consequence of Ehrhart theory. So Theorem 2.5 can be viewed as an easy application of Ehrhart theory when the arrangement is linear. However, if the hyperplane arrangement is not linear, we can not find a way at the moment to interpret Theorem 2.5 as a consequence of Ehrhart theory or Beck and Zaslavsky's reciprocity theorem. We use an easy example to show the reasons. Let the arrangement $\mathcal{A}$ in $\mathbb{R}^2$ consist of three hyperplanes, $H_1 : x = 0, H_2 : y = 0$, and $H_3 : x + y = 1$. Then the polytope $\Delta \in \mathscr{R}(\mathcal{A}, q)$ bounded by H_1, H_2, and H_3 is not dilated as q changes. So Ehrhart theory can not be applied to Δ.

We next prove Theorem 2.5, without the hypothesis of linearity, by applying the convolution formula in Proposition 2.4. For $\boldsymbol{x} \in \mathbb{R}^n$, use $\mathcal{A}_{\boldsymbol{x}}$ to denote the collection of all possible hyperplanes $H(kq)$ who pass through $\boldsymbol{x}$, i.e.,

$$\mathcal{A}_{\boldsymbol{x}} = \{H(kq) \mid H \in \mathcal{A}, k \in \mathbb{Z}, \boldsymbol{x} \in H(kq)\}.$$

Then $\mathcal{A}_{\boldsymbol{x}}$ is a central hyperplane arrangement in $\mathbb{R}^n$. Recall that $\mathscr{R}(\mathcal{A})$ denotes the collection of regions in $\mathbb{R}^n$ separated by all hyperplanes $H \in \mathcal{A}$, and $r(\mathcal{A}) = \big|\mathscr{R}(\mathcal{A})\big|$.

Lemma 2.6. *With previous notations, we have*

$$\#\{\Delta \in \mathscr{R}(\mathcal{A},q) \mid \boldsymbol{x} \in \bar{\Delta}\} = r(\mathcal{A}_{\boldsymbol{x}}), \quad \forall \boldsymbol{x} \in \mathbb{R}^n.$$

Proof. We shall prove it by constructing a bijection $\psi : \{\Delta \in \mathscr{R}(\mathcal{A},q) \mid \boldsymbol{x} \in \bar{\Delta}\} \to \mathscr{R}(\mathcal{A}_{\boldsymbol{x}})$ for any $\boldsymbol{x} \in \mathbb{R}^n$. It is obvious that $M(\mathcal{A},q) \subseteq M(\mathcal{A}_{\boldsymbol{x}})$. Then for all $\Delta \in \mathscr{R}(\mathcal{A},q)$ and $R \in \mathscr{R}(\mathcal{A}_{\boldsymbol{x}})$, we have either $\Delta \in R$ or $\Delta \cap R = \emptyset$, since Δ and R are connected components of $M(\mathcal{A},q)$ and $M(\mathcal{A}_{\boldsymbol{x}})$ respectively. According to $\sqcup_{\Delta\in\mathscr{R}(\mathcal{A},q)}\Delta = M(\mathcal{A},q) \subseteq M(\mathcal{A}_{\boldsymbol{x}}) = \sqcup_{R\in\mathscr{R}(\mathcal{A}_{\boldsymbol{x}})}R$, we can conclude that each $\Delta \in \mathscr{R}(\mathcal{A},q)$ is contained in a unique region $R \in \mathscr{R}(\mathcal{A}_{\boldsymbol{x}})$, denoted R_Δ. It defines the map $\psi : \Delta \mapsto R_\Delta$. To prove ψ is surjective, consider the set $\Sigma_R = \{\Delta \in \mathscr{R}(\mathcal{A},q) \mid \Delta \subseteq R\}$ for any $R \in \mathscr{R}(\mathcal{A}_{\boldsymbol{x}})$. Note that $R \cap \Delta = \emptyset$ for all $\Delta \in \mathscr{R}(\mathcal{A},q) - \Sigma_R$, and R is an open set. Thus $R \cap \big(\cup_{\Delta\in\mathscr{R}(\mathcal{A},q)-\Sigma_R}\,\bar{\Delta}\big) = \emptyset$. Since $\cup_{\Delta\in\mathscr{R}(\mathcal{A},q)}\bar{\Delta} = \mathbb{R}^n$, we then obtain

$$R - \cup_{\Delta\in\Sigma_R}\bar{\Delta} = R - \cup_{\Delta\in\mathscr{R}(\mathcal{A},q)}\bar{\Delta} = \emptyset.$$

It implies that $\bar{R} = \cup_{\Delta\in\mathscr{R}(\mathcal{A},q)}\bar{\Delta}$. On the other hand, we obviously have $\boldsymbol{x} \in \bar{R}$ for $R \in \mathscr{R}(\mathcal{A}_{\boldsymbol{x}})$. Thus $\boldsymbol{x} \in \bar{\Delta}$ for some $\Delta \in \Sigma_R$. Namely, each region $R \in \mathscr{R}(\mathcal{A}_{\boldsymbol{x}})$ contains a region $\Delta \in \mathscr{R}(\mathcal{A},q)$ such that $\boldsymbol{x} \in \bar{\Delta}$, which proves the surjection of ψ. To show that ψ is injective, suppose that we have $\Delta_1, \Delta_2 \in \{\Delta \in \mathscr{R}(\mathcal{A},q) \mid \boldsymbol{x} \in \bar{\Delta}\}$ and $\Delta_1 \neq \Delta_2$ such that $R = R_{\Delta_1} = R_{\Delta_2}$. Let $H(kq)$ be a separating hyperplane of Δ_1 and Δ_2, i.e., $\Delta_1 \subseteq H(kq)^+$ and $\Delta_2 \subseteq H(kq)^-$ $\big($or, $\Delta_1 \subseteq H(kq)^-$ and $\Delta_2 \subseteq H(kq)^+\big)$, where H^+ and H^- are two closed half space of $\mathbb{R}^n$ divided by H. Then we have $\boldsymbol{x} \in \bar{\Delta}_1 \cap \bar{\Delta}_2 \subseteq H(kq)^+ \cap H(kq)^- = H(kq)$. It implies that $H(kq) \in \mathcal{A}_{\boldsymbol{x}}$. From the assumption $R \in \mathscr{R}(\mathcal{A}_{\boldsymbol{x}})$, we then have $R \cap H(kq) = \emptyset$. Notice that R is connected. So $R \cap H(kq)^+ = \emptyset$ or $R \cap H(kq)^- = \emptyset$. This contradicts to $\Delta_1, \Delta_2 \subseteq R$ and $\Delta_1 \subseteq H(kq)^+, \Delta_2 \subseteq H(kq)^-$, which completes the proof. □

Let $\mathcal{A}$ and $\mathcal{B}$ be two hyperplane arrangements in $\mathbb{R}^n$. We call $\mathcal{B}$ a *translation* of $\mathcal{A}$ if there is a $t_H \in \mathbb{R}$ for each $H \in \mathcal{A}$ such that $\mathcal{B} = \{H(t_H) \mid H \in \mathcal{A}\}$.

Lemma 2.7. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are central hyperplane arrangements. If $\mathcal{B}$ is a translation of $\mathcal{A}$, then $r(\mathcal{B}) = r(\mathcal{A})$.*

Proof. Since $\mathcal{A}$ and $\mathcal{B}$ are central, take $\boldsymbol{a} \in \cap_{H\in\mathcal{A}}H$ and $\boldsymbol{b} \in \cap_{H\in\mathcal{B}}H$. Consider the translation $\tau : \boldsymbol{x} \mapsto \boldsymbol{x} + \boldsymbol{b} - \boldsymbol{a}$ of $\mathbb{R}^n$. Since $\mathcal{B}$ is a translation of $\mathcal{A}$, then τ defines a bijection between $\mathcal{A}$ and $\mathcal{B}$, as well as a bijection between $L(\mathcal{A})$ and $L(\mathcal{B})$. So $r(\mathcal{B}) = r(\mathcal{A})$. □

Given $H \in \mathcal{A}$, let $H_{(q)}$ be a subset of $C(n,q)$ defined by

$$H_{(q)} = C(n,q) \cap \big(\cup_{k\in\mathbb{Z}}\, H(kq)\big).$$

Then $\mathcal{A}_{(q)} = \{H_{(q)} \mid H \in \mathcal{A}\}$ forms an arrangement of sets in $C(n,q)$. Similar as before, we have the notations $L(\mathcal{A}_{(q)})$, $M(\mathcal{A}_{(q)})$, $\mathcal{A}_{(q)}|X_{(q)}$, $\mathcal{A}_{(q)}/X_{(q)}$ for $X_{(q)} \in L(\mathcal{A}_{(q)})$, i.e.,

$$\begin{aligned} &L(\mathcal{A}_{(q)}) = \{\cap_{H\in\mathcal{B}}H_{(q)} \mid \mathcal{B} \subseteq \mathcal{A}\}, && M(\mathcal{A}_{(q)}) = C(n,q) - \cup_{H\in\mathcal{A}}H(q),\\ &\mathcal{A}_{(q)}|X_{(q)} = \{H_{(q)} \mid H \in \mathcal{A}, X \subseteq H\}, && \mathcal{A}_{(q)}/X_{(q)} = \{H_{(q)} \cap X_{(q)} \mid H \notin \mathcal{A}\ X_{(q)}\}. \end{aligned}$$

Consider the map

$$\rho : C(n,q) \to \mathbb{F}_q, \quad \boldsymbol{x} \mapsto \boldsymbol{x} \mod q.$$

It is obvious that ρ is a bijection. Moreover, for any $\mathcal{B} \subseteq \mathcal{A}$, we have $\rho\big(\cap_{H\in\mathcal{B}}\,H_{(q)}\big) = \cap_{H\in\mathcal{B}}H_q$. So ρ automatically induces an isomorphism of $L(\mathcal{A}_{(q)})$ and $L(\mathcal{A}_q)$ with $\rho(X_q) = X_{(q)}$. It is easy to see that $\big|X_q\big| = \big|X_{(q)}\big|$, and then $\big|M(\mathcal{A}_q)\big| = \big|M(\mathcal{A}_{(q)})\big|$.

Proof of Theorem 2.5: Since that

$$M\big(\mathcal{A}_{(q)}/X_{(q)}\big) = \cap_{H\in\mathcal{A}|X}H_{(q)} - \cap_{H\notin\mathcal{A}|X}H_{(q)},$$

then we have $C(n,q) = \sqcup_{X\in L(\mathcal{A})}M\big(\mathcal{A}_{(q)}/X_{(q)}\big)$. For any $\boldsymbol{x} \in M\big(\mathcal{A}_{(q)}/X_{(q)}\big)$, we can see that $\mathcal{A}_{\boldsymbol{x}}$ is a translation of $\mathcal{A}|X$. Since both $\mathcal{A}_{\boldsymbol{x}}$ and $\mathcal{A}|X$ are central, it follows by Lemma 2.7 that $r(\mathcal{A}_{\boldsymbol{x}}) = r(\mathcal{A}|X)$. Applying Lemma 2.6, we have

$$\sum_{\Delta\in\mathscr{R}(\mathcal{A},q)} \big|\bar{\Delta} \cap C(n,q)\big| = \sum_{\boldsymbol{x}\in C(n,q)} \#\{\Delta \mid \boldsymbol{x} \in \bar{\Delta}\} = \sum_{\boldsymbol{x}\in C(n,q)} r(\mathcal{A}_{\boldsymbol{x}}) = \sum_{X\in L(\mathcal{A})} r(\mathcal{A}|X)\big|M\big(\mathcal{A}_{(q)}/X_{(q)}\big)\big|.$$

Since $\left|M\left(\mathcal{A}_{(q)}/X_{(q)}\right)\right| = \left|M\left(\mathcal{A}_q/X_q\right)\right|$, it follows that

$$\bar{\chi}(\mathcal{A},q) = \sum_{X\in L(\mathcal{A})} r(\mathcal{A}|X)\left|M\left(\mathcal{A}_q/X_q\right)\right|,$$

which completes the proof by Proposition 2.4.

3 Convolution Formulae on Tutte Polynomials

In this section, we shall apply Theorem 1.1 to formulate those convolution identities mentioned in [4,6]. Let M be a matroid with the ground set E and the rank function r_M. For simplicity, write $r_M(E) = r(M)$ for the rank of the matriod M. The rank generating function $R_M(x,y)$ and the Tutte polynomial $T_M(x,y)$ of M are defined by

$$R_M(x,y) = \sum_{A\subseteq E} x^{r(M)-r_M(A)}\, y^{|A|-r_M(A)}, \quad T_M(x,y) = R_M(x-1,y-1).$$

If S is a subset of E, the restriction of M to S, written as $M|S$, is the matroid on the ground set S whose rank function is $r_{M|S}(A) = r_M(A)$ for all $A \subseteq S$. If T is a subset of E, the contraction of M by T, written as M/T, is the matroid on the ground set $E - T$ whose rank function is $r_{M/T}(A) = r_M(A\cup T) - r_M(T)$ for all $A \subseteq E - T$. With these definitions, we have

$$\begin{aligned} R_{M|S}(x,y) &= \sum_{A\subseteq S} x^{r(M|S)-r_M(A)}\, y^{|A|-r_M(A)}; \\ R_{M/T}(x,y) &= \sum_{T\subseteq A\subseteq E} x^{r(M)-r_M(A)}\, y^{|A|-|T|-r_M(A)+r_M(T)}. \end{aligned}$$

To write the Tutte polynomial as the Möbius conjugation, consider the poset $(2^E,\subseteq)$ whose Möbius function is given by $\mu(A,B) = (-1)^{|A-B|}$ for all $A\subseteq B\subseteq E$. Let

$$f(A) = (-x)^{r(M)-r_M(A)}, \quad g(A) = (-y)^{|A|-r_M(A)}, \quad \forall\, A\subseteq E.$$

Then we have

$$\mu^*(fg)(\emptyset,E) = \sum_{A\subseteq E} \mu(\emptyset,A)\,(-x)^{r(M)-r_M(A)}\,(-y)^{|A|-r_M(A)} = (-1)^{r(M)} R_M(x,y).$$

Similarly, we can obtain

$$\begin{aligned} (-1)^{r(M|S)} R_{M|S}(-1,y) &= \mu^*(g)(\emptyset,S), \\ (-1)^{r(M/T)} R_{M/T}(x,-1) &= \mu^*(f)(T,E). \end{aligned}$$

where the poset $(2^{E-T},\subseteq)$ for the last identity is identified with the interval $[T,E]$ of the poset $(2^E,\subseteq)$. Since $r(M) = r(M|A) + r(M/A)$ for all $S\subseteq E$ and $\mu^*(fg) = \mu^*(g) * \mu^*(f)$ by Theorem 1.1, we obtain

$$R_M(x,y) = \sum_{A\subseteq E} R_{M|A}(-1,y)\, R_{M/A}(x,-1).$$

This is equivalent to the main result obtained by W. Kook, V. Reiner, and D. Stanton [4].

Theorem 3.1. [4] *The Tutte polynomial $T_M(x,y)$ satisfies that*

$$T_M(x,y) = \sum_{A\subseteq E} T_{M|A}(0,y)\, T_{M/A}(x,0).$$

Similar method can be applied to obtain the convolution identities of the subset-corank polynomial $SC_M(\boldsymbol{x},\lambda)$ defined in [6]. Denote by x_e the indeterminate indexed by $e\in E$. Given $A\subseteq E$,

write x_A for the monomial $\Pi_{e\in A} x_e$ and $\boldsymbol{x}$ for the collection of x_e for all $e \in E$. The subset-corank polynomial $SC_M(\boldsymbol{x}, \lambda)$ is defined to be

$$SC_M(\boldsymbol{x}, \lambda) = \sum_{A\subseteq E} x_A \, \lambda^{r(M)-r_M(A)}.$$

From the definition, for any $S, T \subseteq E$, we have

$$\begin{aligned} SC_{M|S}(\boldsymbol{x}, \lambda) &= \sum_{A\subseteq S} x_A \, \lambda^{r(M|S)-r_M(A)}, \\ SC_{M/T}(\boldsymbol{x}, \lambda) &= \sum_{T\subseteq A\subseteq E} x_{A-T} \, \lambda^{r(M)-r_M(A)}. \end{aligned}$$

Take $(2^E, \subseteq)$ to be the poset and let

$$f(A) = (-x)_A = (-1)^{|A|} x_A, \quad g(A) = \lambda^{r(M)-r_M(A)}, \quad \forall\, A \subseteq E.$$

Then we can easily obtain

$$\mu^*(fg)(\emptyset, E) = SC_M(\boldsymbol{x}, \lambda).$$

On the other hand, for any $S, T \subseteq E$, we have

$$\begin{aligned} \mu^*(f)(\emptyset, S) &= \sum_{A\subseteq S} x_A = (x+1)_A, \\ \mu^*(g)(T, E) &= \sum_{T\subseteq A\subseteq E} (-1)^{|A-T|} \, \lambda^{r(M)-r_M(A)} = SC_{M/T}(\boldsymbol{-1}, \lambda). \end{aligned}$$

So we have the following formula which is the identity 5 obtained in [6].

Theorem 3.2. [6] *The subset-corank polynomial $SC_M(\boldsymbol{x}, \lambda)$ satisfies*

$$SC_M(\boldsymbol{x}, \lambda) = \sum_{A\subseteq E} (x+1)_A \, SC_{M/T}(\boldsymbol{-1}, \lambda).$$

Similarly, the identity 1 in [6] can be easily obtained in this way. Let

$$f(A) = x_A \, \lambda^{r(M)-r_M(A)}, \quad g(A) = (-y)_A \, \xi^{r(M)-r_M(A)}, \quad \forall\, A \subseteq E.$$

Then we have

$$\mu^*(fg)(\emptyset, E) = SC_M(\boldsymbol{xy}, \lambda\xi),$$

Where $\boldsymbol{xy}$ means the collection of $x_e y_e$ for all $e \in E$. On the other hand, for any $S, T \subseteq E$, we have

$$\begin{aligned} \mu^*(f)(\emptyset, S) &= \sum_{A\subseteq S} (-1)^{|A|} \, x_A \, \lambda^{r(M)-r_M(A)} = \lambda^{r(M)-r(M|S)} \, SC_{M|S}(-\boldsymbol{x}, \lambda), \\ \mu^*(g)(T, E) &= \sum_{T\subseteq A\subseteq E} (-1)^{|A-T|} \, (-y)_A \, \xi^{r(M)-r_M(A)} = (-y)_T \, SC_{M/T}(\boldsymbol{y}, \xi). \end{aligned}$$

Applying Theorem 1.1, we have

Theorem 3.3. [6] *The subset-corank polynomial $SC_M(\boldsymbol{x}, \lambda)$ satisfies*

$$SC_M(\boldsymbol{xy}, \lambda\xi) = \sum_{A\subseteq E} \lambda^{r(M)-r(M|S)} \, (-y)_T \, SC_{M|S}(-\boldsymbol{x}, \lambda) \, SC_{M/T}(\boldsymbol{y}, \xi).$$

We remark that other identities in [6] and [5] can be obtained in a similar way as above three.

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